# Introduction to Electrical Engineering (EE 202)

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# Contents

# §1 Electrical Intuition/Instrumentation

## §1.1 Electrical Intuition

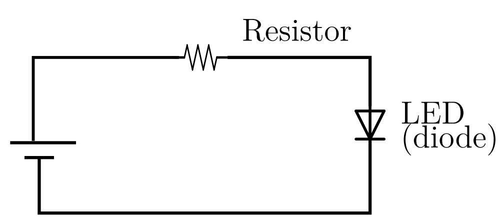
What is electrical engineering?

- Electromagnetics, Chemistry, Physics
- Solid state
- Passives (resistors, capacitors, inductors), transistors, diodes
- Integrated circuit(IC)s
- Schematics
- Code and algorithms (e.g. FFT, ML)

Question: What is electricity and how does it flow

- Electricity is a flow of electrons ($e^-$ )
- Electric potential ($V$) works as a "force" pushing the electricity flow

A **battery** is a source of electric potential. For example, a $1.5V$ battery has a potential difference of $1.5V$ between its positive and negative terminal. For electricity to flow, there should be a *closed loop* between the positive and negative sides.




Assumption: there is no *voltage drop* within a wire

**Definition Kirchhoff's Voltage Law**
In a closed loop, the sum of voltage is 0:

$$\sum_{\text{loop}} V = 0$$

Polarized component: direction matters

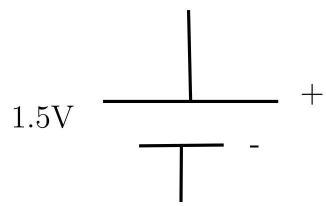
- anode:
- cathode

longer leg has to be on higher potential

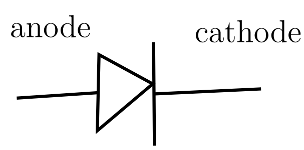
**forward voltage**: a minimum voltage for the LED to light up

we can fix this by connecting two batteries in **series**

The schematic symbol for a battery (longer side is +)




A symbol for LED (diode) is




**Definition diode**
A **diode** only allows the flow of current in one direction

Anode is the side with the higher potential

- A **loop** of a circuit starts and ends at the same point.
- A **node** is a location within the circuit which is at the same potential
- A **junction** of a circuit

**Example:** When I touch two ends of a battery, current flows through my body.
**Yes**, a loop is formed by the two fingers, and there is a finite resistance in our body which makes some current flow.

**Example:** If I connect a AA battery ($1.5V$) to a $1000V$ power wire, the other end of the battery will be at $1001.5V$.
**True**,

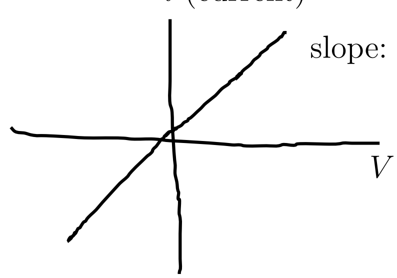
**Example:** If I touch a $1000V$ wire, I'm going to get a nasty shock
**Depends**, becasue if we are grounded, that is going to form a closed loop and going to hurt.

(talk about resistors)

### §1.1.1 Voltage and Current

Current: flow of charges Voltage: pushes the charges

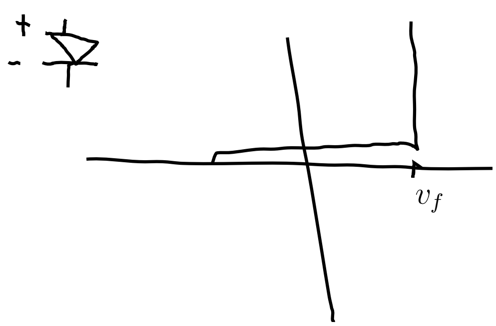
The I-V curve




The slope of $i - V$ curve is the resistance $R$.

Ohm's Law

$$V = iR$$




For diodes, there is a spike of current after a certain voltage. A diode only allows a current to pass in one direction if $V > V_f$.

Resistors in **series**

- Current $I$ stays the same across circuit

- Each element may have different voltage drops

Resistors in **parallel**: they should share two nodes to be parallel

- The voltage $V$ must stay the same across circuit
- Each element may have different current

Simplifications for analysis (ideal components)

- wires: no resistance ($R = 0$)
- Resistors: purely linear in the $IV$ curve ($V = IR$)
- Diodes, LEDs: no changing voltage drop due to current
- Sources: ideal (no internal resistance)

## §1.2 Instrumentation

Breadboard

- in the middle, the five nodes are electrically connected
- in the edges, the nodes are for power connection and are horizontally connected all the way through

Analog Discovery Tool

- two oscilloscopes: measure voltage over time
- ground reference: voltage is always relative
- power supply
- W1 and W2 are two independent waveform generator
- Digital logic:

**Potentiometers**: variable resistance (turn the knob)

The second pin acts as a 'wiper' and changes the resistance between pin 1 and 2.

## §1.3 Voltage, Current, Ohms Law, Sources

### §1.3.1 Coulombs Law and Electric Field

**Definition Coulomb's Law**
Given charges $q_1$ and $q_2$, the force between two objects are

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r} = E_{12} q_2 \hat{r}$$

The term $\mathbf{E}_{12}$ is the electric field generated by the first charge.

- Electrons: $q = -1.602 \times 10^{-19}$ C.
- Protons: $q = 1.602 \times 10^{-19}$ C.

### §1.3.2 Current and Electric Potential

**Definition Current**
Current is the rate of change of charge with respect to time

$$i = \frac{dq}{dt}$$

Analogy with gravitational potential: If a mass $(m)$ is raised to a certain height ( $h$ ) relative to the ground ($h = 0$ ), the **gravitational potential energy** is $U = mgh$. In this case, the **absolute potential** is the *mass independent* value, $gh$. The gravitational force on the object is $F = mg$.

For electric potential, the charge $(q)$ is $h$ away from the ground $(V = 0)$. The electric force is $F = qE$.

- potential energy: $qEh$
- absolute potential: $\mathbf{E}h$ (assuming constant, conservative $\vec{E}$)

Zero potential can be chosen arbitrarily (potential is always relative)

### §1.3.3 Energy and power

moving one colomb of charge through a potential of one volt yields one jouls of potential energy

The **conventional current** moves in the *opposite* direction of the electrons. (Think about positive charges moving)

$$\text{Joule} = \text{coulomb} \times \text{Volts}$$

For $i$ amps elevated by $V$ volts, the battery must source $iV$ Watts (joules/sec)

**Definition Power**
Power is current times voltage

$$P(t) = i(t)V(t)$$

The SI unit for power is Watts (W)

When the current flows into the battery, the battery is **absorbing**(*dissipating*) power.

- Generating power: current flows from negative to positive terminal
- Dissipating power: current flows from positive to negative terminal

When the battery flips, the polarity of the resistor also changes because resistor can only dissipate power.

**Definition Ohm's Law**
For resistors,
$$V = iR$$
where $R$ is resistance and is measured in ohms ($\omega$)

For resistors,
$$\begin{aligned} P(t) &= i(t)v(t) \\ &= i(t)\left[i(t)R\right] = i^2(t)R \end{aligned}$$

## §1.3.4 Sources

A DC voltage source has a *fixed voltage* no matter the current. In a i-V graph, voltage source represents a *vertical* line.

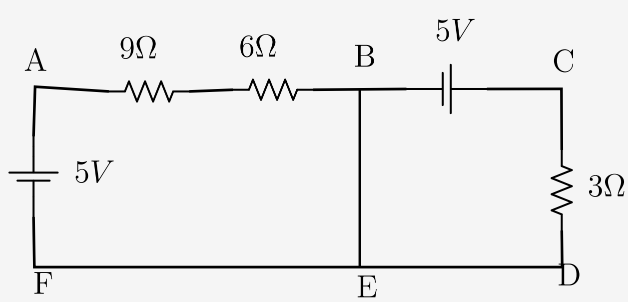
A **current source** supplies a set current no matter what voltage. In a i-V graph, current source represents a *horizontal* line.

For **ideal** sources, we make assumption of no internal resistance

**Definition Kirchhoff's Laws**
**Kirchhoff's Voltage Law**: The sum of all rises and falls of electric potential (voltage) around *any* closed loop in a circuit is 0:
$$\sum_{\text{loop}} \Delta V = 0$$
**Kirchhoff's Current Law**: In a node (junction), the sum of all currents entering the nodes must equal the sum of all currents leaving the node:
$$\sum_{\text{junction}} i = 0$$

The directions of loops can be chosen arbitrarily, and the answer is always correct if calculation stays consistent. If the current returns a negative value, this suggests opposite direction.

**Example: KVL Example**



*Solution.* The polarity can be labled based on the arbitrary current direction we chose.

Loop $ABEF$:

$$\sum \Delta V = 12 - 9i_1 - 6i_1 - 10i_3$$

Loop $BCDE$:

$$\sum V = 5 - 3i_2 + 10i_3$$

Node $B$:

$$i_1 = i_2 + i_3$$

This gives a system of equation with 3 unknowns and 3 equations. Solving,

$$\begin{aligned} i_1 &= 0.916A \\ i_2 &= 1.089A \\ i_3 &= -0.173A \end{aligned}$$

The negative sign on the $i_3$ suggests that the current actually flows *up*.

**Power Analysis**

Components generating power: $12V$ battery, $5V$ battery

- $P_{12V} = iV = (0.916)(12) = 10.987W$
- $P_{5V} = iV = (1.089)(5) = 5.444W$

Components dissipating power: all resistors

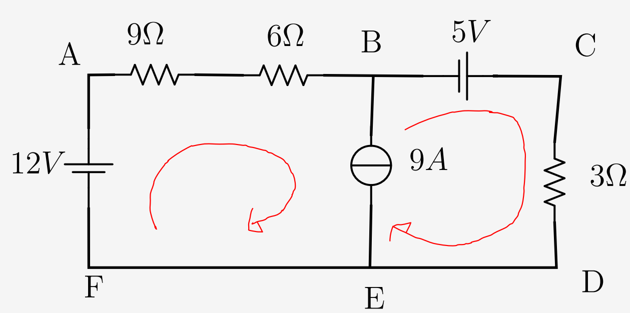
- $P_{9\Omega} = iV = i^2R = (0.916)^2(9) = 7.544W$.

Im too lazy to write rest but power generated should equal power dissipated

$$P_{gen} = P_{dis} = 16.413W$$

□

**Example: KVL with a current source**




*Solution.* Loop $ACDF$:

$$12 - 9i_1 - 6i_1 + 5 - 3i_2$$

From node $B$:

$$i_2 = i_1 + 9$$

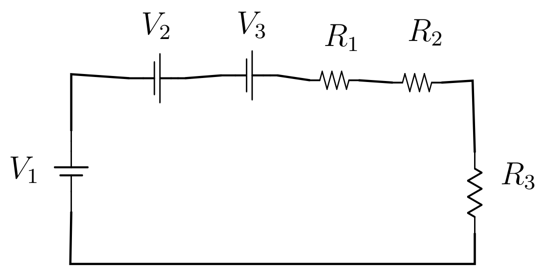
Solving, we get

$$i_1 = -0.556A \qquad i_2 = i_1 + 9 = 8.444A$$

Now we need to solve for the **voltage over the current source**. □

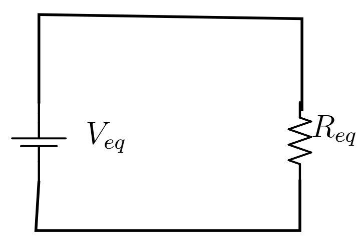
If the $3\Omega$ resistor becomes an open circuit,

## §1.4 Circuit Simplification




$$V_1 + V_2 + V_3 = i(R_1 + R_2 + R_3)$$

Here, we can say that $V_1 + V_2 + V_3$ is an equivalent voltage $V_{eq}$ and $R_1 + R_2 + R_3$ as the equivalent resistance $R_{eq}$. This reduces to a circuit



# §2 Circuit Analysis

## §2.1 Voltage, Current, Ohm's Law

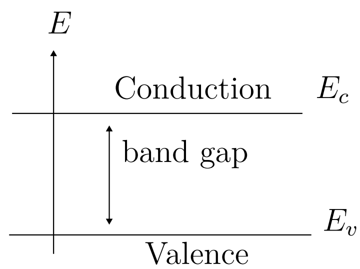
## §2.2 Kirchhoff's Laws

# §3 Semiconductors

## §3.1 PN Junctions

### §3.1.1 Intrinsic Semiconductor

Electrons have multiple shells. The electron "jumps" to an upper shell as it gains energy.




If an electron moves out, it leaves a "vacancy space" with a much lower energy space. Then, another electron moves to fill in the space. This shows a pattern of conduction:

- Electrons are moving down the lattice
- vacancies are "moving up" the lattice

For metals, conduction and valence bands overlap, making them good conductors.

### §3.1.2 Extrinsic Semiconductor

We 'dope' intrinsic semiconductors

- **N-type**: dope with donors (atoms with one more valance electron than intrinsic semiconductor)
- **P-type**: dope with acceptors (atoms with one less valence electron)

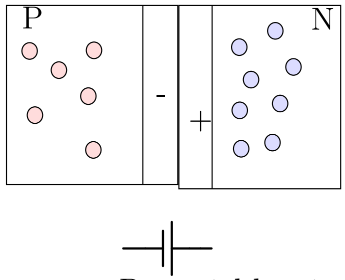
**What is doping?**: Process of bombarding the material with other atoms, and displacing intrinsic atoms

In n-type, majority carriers are electrons, and in p-type, majority carriers are the holes.

The material is still **net neutral**. They are not ions

#### PN Junction

Red circles are holes, blue circles are electrons




Potential barrier

**Depletion region**: there is a barrier in the middle depleted of charge carriers

We have to overcome this potential barrier if we want the material to conduct.

**Built-in voltage**

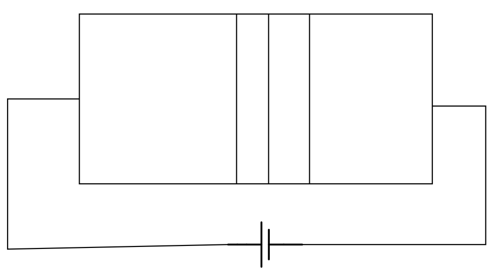
$$V_{bi} = V_T \ln\left(\frac{N_A N_D}{n_i^2}\right)$$

where $V_T$ is the thermal voltage, $N_A$ is the acceptor concentration on p-side, $N_D$ is the donor concentration on the $n$ side, and $n_i$ is intrinsic carrier concentration.

$V_T$ is a constant just dependent on room temperature. $n_i$ is a constant dependant on material and temperature.

**External Bias**

We have a PN junction connected to an external voltage source



- If $V < 0$, depletion region increases and no current flow (reverse bias)
- If $0 < V < V_{bi}$, depletion region decreases but no significant current flow (reverse bias)
- If $V_{bi} < V$, we overcome potential barrier and current flows (forward bias)

(drift vs diffusion) There is a diffusion current whenever there is a difference in concentration (because they want low energy)

Electrons are going to rush into the P side –> diffusion current

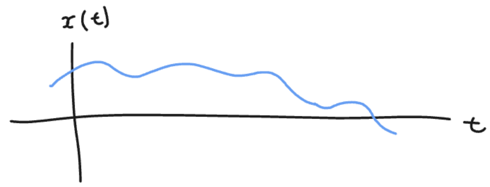
The ionization creates an electric field

# §3.2 Signals and Systems

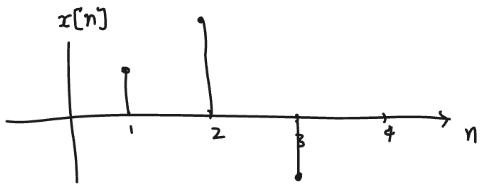
## §3.2.1 Signals

**Definition Signal**
Signals are anything that conveys information
- often converted to a voltage
- can be multi-dimensional

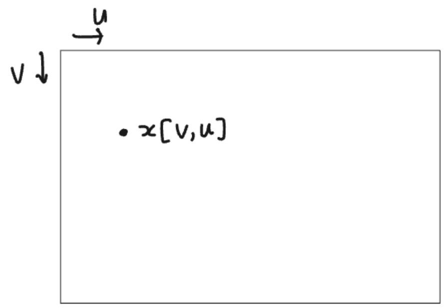
We can represent continuous signals with $x(t)$ :




Discrete signals are $x\left[n\right]$




For two dimensional signals,

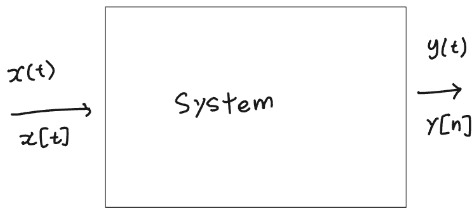



Why do we use signals?

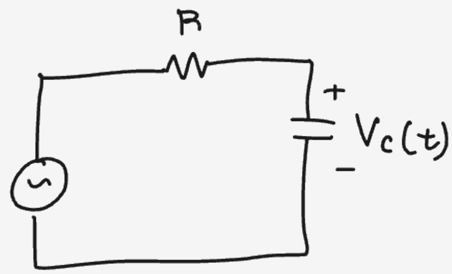
- we can analyze and predict things (eg. stock market)
- we can transform (eg. image compression)

## §3.2.2 Systems

For a system, there is a signal coming in and a signal coming out (can be either discrete or continuous)




**Example: Continuous time system**




Solving in time domain,

$$
\begin{aligned}
V_s(t) &= V_R(t) + V_C(t) \\
&= i(t)R + V_C(t) \\
&= C\frac{dV_C(t)}{dt}R + V_C(t)
\end{aligned}
$$

This circuit is a system that converts $V_S(t)$ to $V_C(t)$ in the form

$$\boxed{\frac{dV_c(t)}{dt} + \frac{1}{RC}V_C(t) = \frac{1}{RC}V_s(t)}$$

**Example: Discrete time system**
For a bank account,

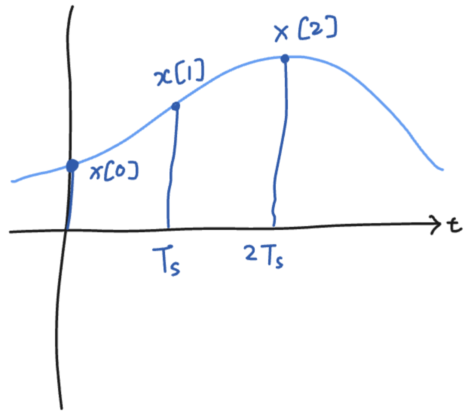
- $y[n]$ : balance at month $n$
- $x[n]$ : deposit at month n
- 5% interest rate

The system is

$$y[n] = 1.05y[n-1] + x[n]$$

A **sampler** takes a continuous signal $x(t)$ and outputs a discrete signal $x[n]$

We would get a signal output of $x\left(nT_s\right)$. where $T_s$ is the sampling period.




The signals are sampled in uniform sampling periods. We are grabbing integer values of the continuous signals

How fast do should we sample? We have to make an assumption that we are going to draw the *lowest frequency signals* that goes through the sample.

> Any signal can be a summation (linear combinations) of cosines

Corollary: if one sampling rate is fast enough, then the faster sampling rate is also fast enough

How often must we sample a pure sinusoid?

- For once per period, there will be a straight line and it is not enough
- For twice per period, we can know the frequency but we dont know the amplitude and phase
- For three times per period,

sampling frequency must be greater than at least twice faster than the highest frequency in the signal

> **Example: Music**
> Music is often sampled at 44100 hz because humans can hear around up to 22000 hz

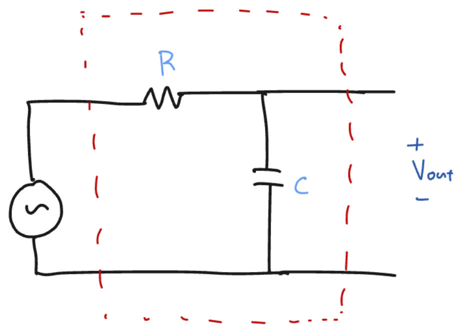
If the sampling frequency is too low, there is a problem

**aliasing**

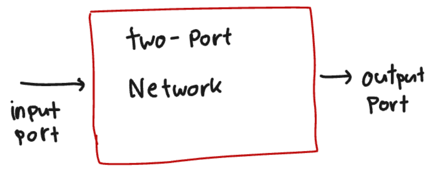
# §4 Phasor

## §4.1 Two Port Network

Consider a circuit




There is an **input** voltage and an **output** voltage. This RC circuit can be characterized as **two-port network** where




We can figure out $V_{out}$ with phasor analysis:

$$\tilde{V_{out}} = \tilde{V}_{in}\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \tilde{V}_{in}\frac{1}{1 + j\omega RC}$$

Then, we can say that

$$H(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1}{1 + j\omega RC}$$

This makes $H(\omega)$ likes a function, where $\tilde{V}_{out} = H(\omega)\tilde{V}_{in}$.

> **Definition Transfer function**
> This $H(\omega)$ is called a **transfer function**

The amplitude of the transfer function ignores the phases

$$|H(\omega)| = \left|\frac{\tilde{V}_{out}}{\tilde{V}_{in}}\right| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

Then,

$$\measuredangle H(\omega) = \measuredangle\tilde{V}_{out} - \measuredangle\tilde{V}_{in}$$

## §4.2 Filtering

A filter passes some frequency and cuts off other. We rewrite the magnitude of $H(\omega)$ in terms of $f$ where $\omega = 2\pi f$

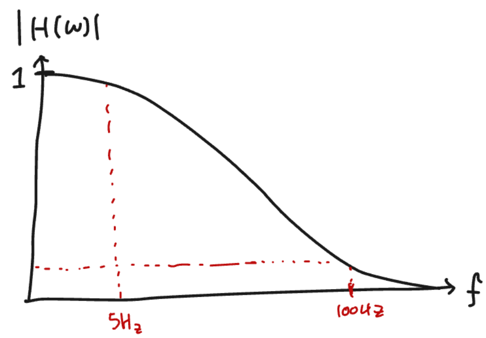
$$|H(f)| = \frac{1}{\sqrt{1+(2\pi fRC)^2}} = \frac{1}{\sqrt{1+\left(\frac{f}{B}\right)^2}}$$

where $B = \frac{1}{2\pi RC}$ and is called the **bandwidth** of the filter.

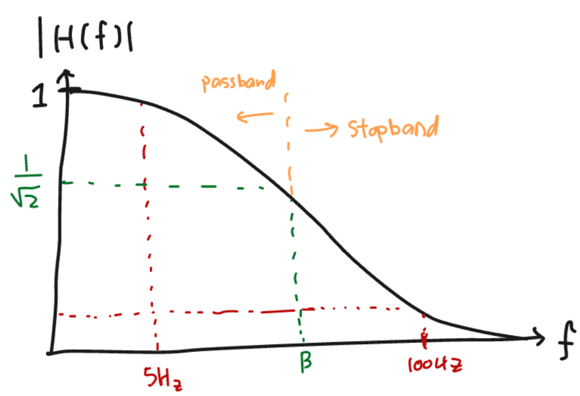
If we write the phase,

$$\measuredangle H(f) = -\tan^{-1}(2\pi fRC) = -\tan^{-1}\left(\frac{f}{B}\right)$$

Then, we can plot the filter




We can see that low frequency signals return a similar value, but high frequency signal gets **attenuated**. This is called a **Low Pass Filter (LPF)** since it only lets low frequencies through.

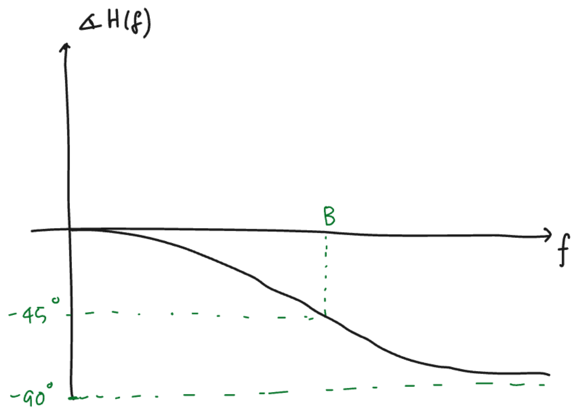



The bandwidth $B$ is the point where the amplitude drops below $\frac{1}{\sqrt{2}}$. The frequencies greater than bandwidth is called a **stopband** and frequencies lower than bandwidth is called **passband**. The bandwidth is often referred as the **cutoff frequency** since it cuts off (not fully) the frequency.

Intuitively, why does the RC circuit act as a low-pass filter (why does it only pass low frequency)
- capacitor has a low impedance at low frequency

Also note that ideal low pass filter would be a square

If we graph the phase,



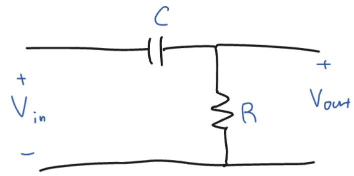

- At $B$, the phase is $-45°$
- The phase approaches $-90°$
- note that this is arctan

For example, lets say

$$V_{in} = 10\cos(2\pi 100t + 50°)$$
$$V_{out} = 1\cos(2\pi 100t - 30°)$$

The amplitude of $V_{out}$ is lower because of attenuation in high frequency, and the phase is also lower

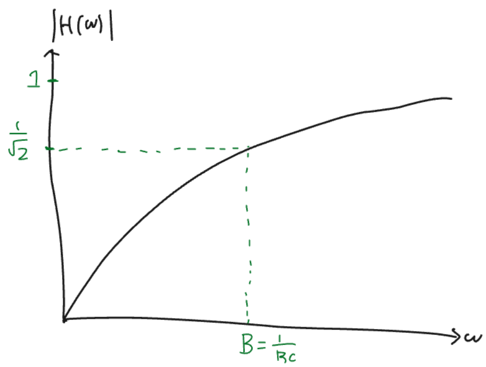
Now, lets look at a **high pass filter**



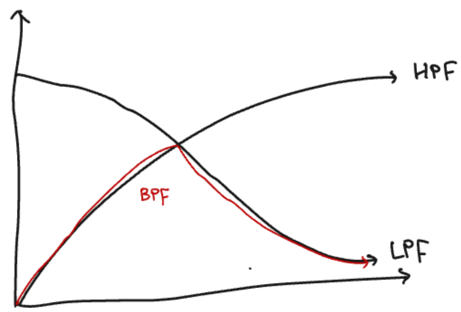

Calculating the transfer function,

$$H(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{R}{R + \frac{1}{j\omega C}}$$

Graphing the amplitude,

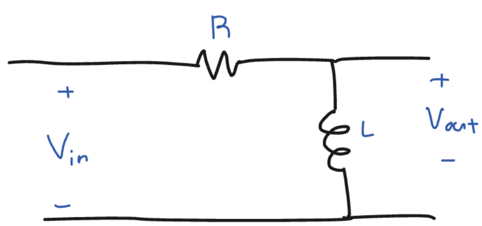



If we put a low pass filter and high pass filter in **series**,




This is called a **band-pass filter**.

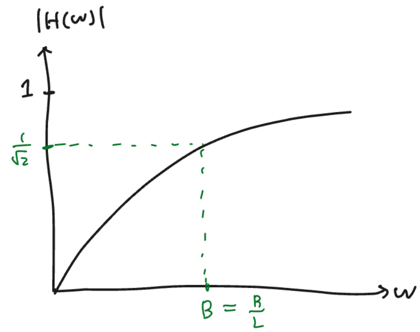
Lets look at another filter




The transfer function is

$$H(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{j\omega L}{R + j\omega L}$$

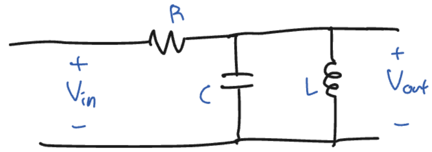
This would also be a high pass filter




It is not a coincidence that

$$B = \frac{1}{\tau} \qquad \tau = \frac{L}{R}$$

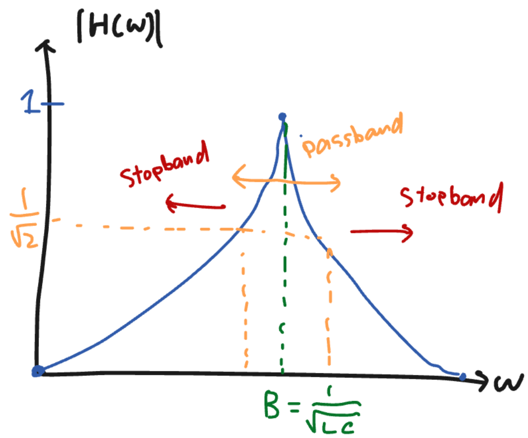
Examples of Band-Pass Filter




The transfer function is

$$H(\omega) = \frac{Z_{LC}}{R + Z_{LC}}$$

- At low frequencies, capacitor is an open circuit and inductor is a short circuit.
- A high frequencies, capacitor acts as an wire but inductor is open
- At resonance, the combination acts like an open circuit (impedance goes to infinity)



# §5 Signals and Systems

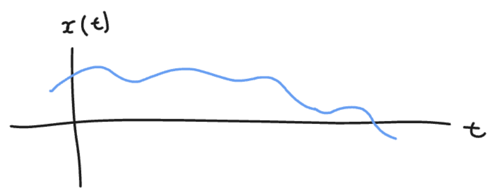
## §5.1 Intro to Signals and Systems

### §5.1.1 Signals

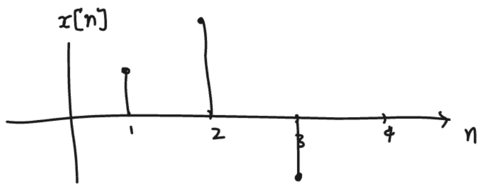
**Definition Signal**
Signals are anything that conveys information
- often converted to a voltage
- can be multi-dimensional (audio, image, machine learning, etc)

Signals can be represented as **continuous** signals or **discrete** signals:

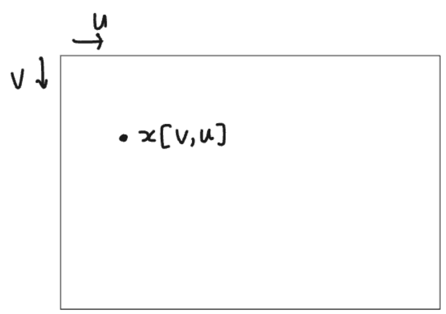
Continuous signals with $x(t)$ :




Discrete signals are $x[n]$:




For two dimensional signals,



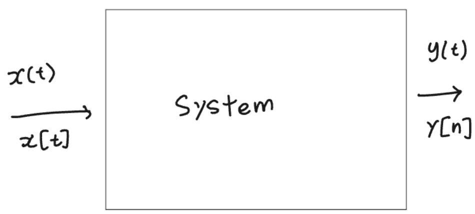

**Question**: Why do we use signals?

- we can analyze and predict things (eg. stock market)
- we can transform (eg. image compression)

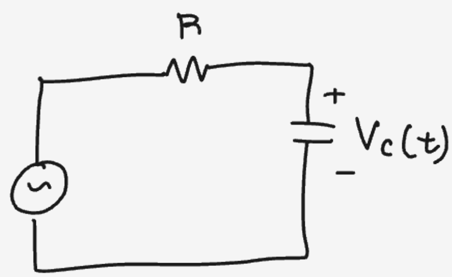
## §5.1.2 Systems

**Definition System**
A system is anything that acts on a signal.

For a system, there is a signal coming in and a signal coming out (can be either discrete or continuous)




**Example: Continuous time system**
Consider a RC circuit




This is a system that converts $V_S$ into $V_C$.

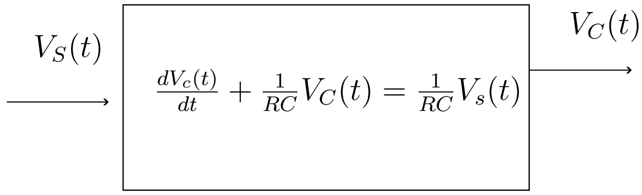
Here, the input voltage $V_S(t)$ is the *input signal* and the output voltage across the capacitor $V_C(t)$ is the *output signal*. We can solve $V_C$ as a function of $V_S$. Solving in time domain,

$$\begin{aligned} V_s(t) &= V_R(t) + V_C(t) \\ &= i(t)R + V_C(t) \\ &= C\frac{dV_C(t)}{dt}R + V_C(t) \end{aligned}$$

What we are left is with a differential equation

$$\frac{dV_C(t)}{dt} + \frac{1}{RC}V_C(t) = \frac{1}{RC}V_S(t).$$

This circuit is a system that converts $V_S(t)$ to $V_C(t)$ in the form




In later classes, this proper analysis technique is called *laplace transform*.

**Example: Discrete time system**
For a bank account,
- $y[n]$ : balance at month $n$
- $x[n]$ : deposit at month n
- 5% interest rate

The system is

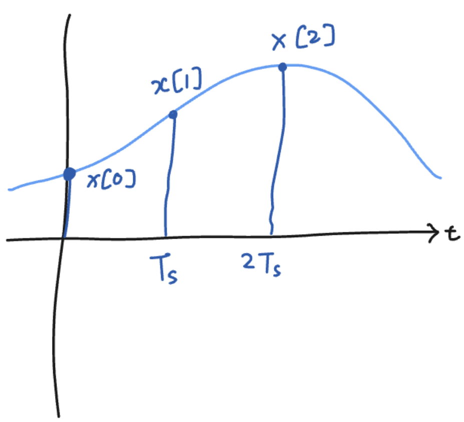
$$y[n] = 1.05y[n-1] + x[n]$$

This is called *difference* equation (instead of differential, which is continuous) which describes the system.

### §5.1.3 Sampling

A **sampler** takes a continuous signal $x(t)$ in and outputs a discrete signal $x[n]$

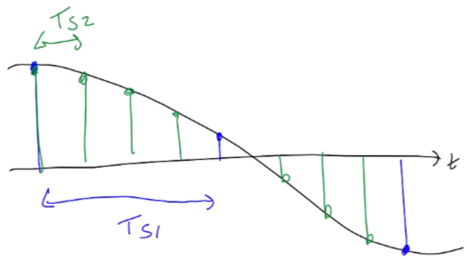

We would get a signal output of $x[n] = x(nT_s)$ where $T_s$ is the sampling period.



This is called **uniform sampling** because the signals are sampled in uniform sampling periods. A sampler essentially grabs integer values of the continuous signals

**Question**: How fast should we sample? There are infinitely many signals that goes through the same discrete points.
**Answer**: We have to make an assumption that we are going to draw the *lowest frequency signals* that goes through the sample.

Lets say we have a sampling period of $T_1$, which is sufficient to sample the following wave:




If $T_{s1}$ is dense enough, then $T_{s2}$ which is $T_{s2} < T_{s1}$ is also dense enough.

**Note**: Any signal can be a summation (linear combinations) of cosines

Corollary: if one sampling rate is fast enough, then the faster sampling rate is also fast enough

How often must we sample a pure sinusoid?

- For once per period, there will be a straight line and it is not enough
- For twice per period, we can know the frequency but we dont know the amplitude and phase
- For three times per period, we have the info

But because we are interested in periods, we sample twice per period (ask this again in OH or smth)

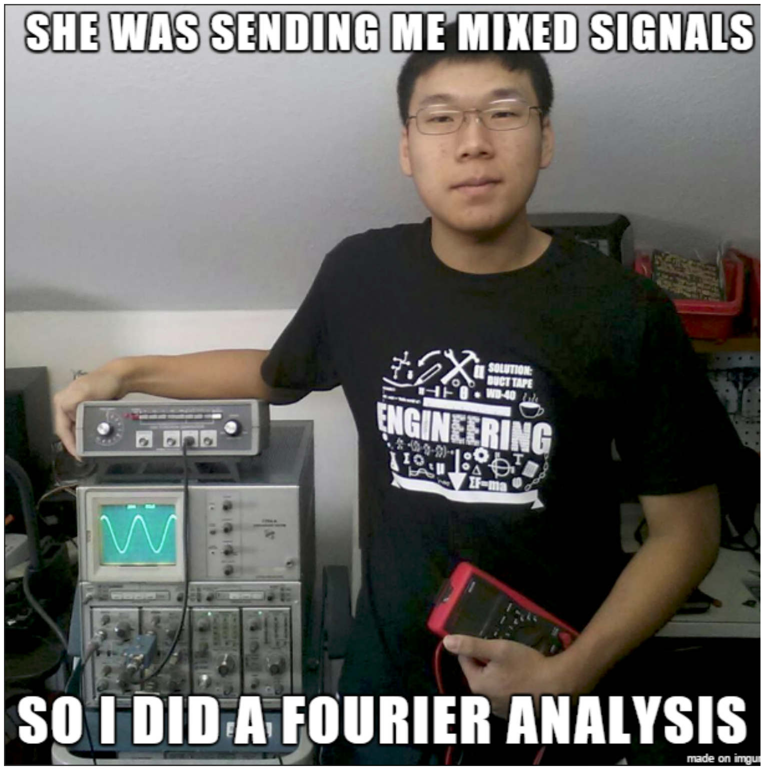
sampling frequency must be greater than at least twice faster than the highest frequency in the signal

**Example: Music**
Music is often sampled at 44100 hz because humans can hear around up to 22000 hz

If the sampling frequency is too low, there is a problem called **aliasing**. This would be covered later in the class.

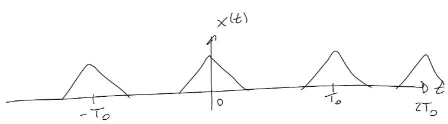
## §5.2 Fourier Series

**Question**: How do we found out frequencies that makes up a signal?




### §5.2.1 Periodicity

A signal repeats every period. Consider a wave




This signal can be written as

$$x(t) = x(t - kT_0)$$

where $k = 0, 1, 2, \ldots$

Any signal can be decomposed into a summation of sine waves, where each waves have individual frequency. The **fundamental** frequency is how often the total signal repeats.

**Definition Fundamentals**
The fundamental is the frequency or the period where the overall signals repeat

- $T_0$ is the fundamental period
- $f_0$ is the fundamental frequency

$$f_0 = \frac{1}{T_0}$$

This same notion of periodicity can be applied to discrete signals

$$x[n] = x[n - kN_0]$$

**Question**: How do we find the fundamental frequency and period?

Lets say there is a signal

$$x(t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t) + \cdots + A_n \cos(2\pi f_n t)$$

how can we get the overall periodicity of $x(t)$?

The **fundamental period** $T_0$ can be stated as the *least common multiple* as constituent periods

$$T_0 = \text{lcm}\,(T_1, T_2, \ldots, T_n)$$

The **fundamental frequency** is the greatest common factor of constituent frequencies

$$f_0 = \frac{1}{T_0} = \text{gcd}\,(f_1, f_2, \ldots, f_n)$$

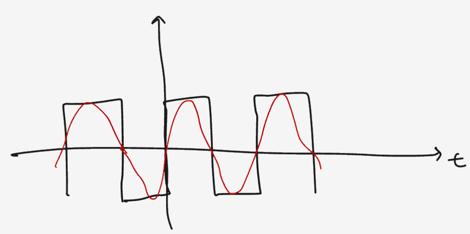
### §5.2.2 Fourier Series

**Big Picture**: **Fourier Series** is a decomposition into parts that are easier to analyze.

- Analysis: breakdown of signal
- Synthesis: combine signals

For fourier series, we want to break our signal up into simple parts (sinusoids)

**Example: Signals with fixed period of 1**




Here, we can analyze a signal with period 1 instead of arbitraty $T_0$.

We can use $\sin(2\pi t)$ or $\cos(2\pi t)$ since they have a frequency of 1.

Observe that $\sin(4\pi t)$ has a frequency of 2Hz and a period of $T = 0.5s$, but *also* repeats every $1s$. In a similar sense, $\sin(6\pi t)$ has a period of $T = \frac{1}{3}s$ but *also* repeats every $1s$, not breaking the periodicity (because gcd of the frequencies are still 1). In this sense, $\sin(2\pi n t)$ where $n \in \mathbb{Z}$ would not break the periodicity of 1.

Revisiting the sampling frequency, we can

1. decompose the waves
2. choose the wave with fastest frequency
3. sample with twice the fastest frequency

periodicity does NOT depend on amplitudes or phases

Using these characteristics, we can express any signal as a summation of sinusoidal waves:

$$x(t) = \sum_{k=1}^{N} A_k \sin(2\pi kt + \phi_k)$$

This is the case for signals with fundamental frequency of 1. $A_k$ and $\phi_k$ are different for each wave because periodicity doesnt depend on amplitudes and phases.

Recall trigonometric identity:

$$\sin(2\pi kt + \phi_k) = \sin(2\pi kt)\cos(\phi_k) + \cos(2\pi kt)\sin(\phi_k)$$

Then, the equation above can be written as

$$\sum_{k=1}^{N} a_k \cos(2\pi kt) + b_k \sin(2\pi kt)$$

The phases $A_k$ got absorbed into the coefficients $a_k$ and $b_k$ where

$$b_k = A_k \cos(\phi_k) \qquad a_k = A_k \sin(\phi_k)$$

To generalize this, we can add the zeroth term:

$$a_0 + \sum_{k=1}^{N} a_k \cos(2\pi kt) + b_k \sin(2\pi kt)$$

In reality, this $a_0$ can be something like a DC offset. Using Euler's formula, we can obtain the final form

Our final form is

$$x(t) = \sum_{k=-N}^{N} c_k e^{j2\pi kt}$$

where the complex value $c_k$ is a **fourier coefficient**

Important thing is that everything that we did here only applies to **periodic signals**. The *fourier transform* addresses this limitation.

**Question**: why are there negative components? (why does $k$ start from $-N$ )

Revisitng the equation,

$$\sum_{k=-N}^{N} c_k e^{j2\pi kt} = \sum_{k=-N}^{N} c_k \cos(2\pi kt) + jc_k \sin(2\pi kt)$$

We know that this also has to equal our sine and cosine counterpart:

$$a_0 + \sum_{k=1}^{N} a_k \cos(2\pi kt) + b_k \sin(2\pi kt)$$

The $k$th frequency is going to equal to

$$a_k \cos(2\pi kt) + b_k \sin(2\pi kt) = c_k \cos(2\pi kt) + c_k \sin(2\pi kt) + c_{-k} \cos(2\pi(-k)t) + jc_{-k} \sin(2\pi(-k)t)$$

Therefore, the $-k$ term is used to **cancel out imaginary values**. Using $\cos(x) = \cos(-x)$ and $\sin(-x) = -\sin(x)$, the whole expression becomes

$$c_k \cos(2\pi kt) + c_{-k} \cos(2\pi kt) + jc_k \sin(2\pi kt) - jc_{-k} \sin(2\pi kt)$$

Therefore,

$$a_k = c_k + c_{-k}$$
$$b_k = jc_k - jc_{-k}$$

We can also solve for $c_k$ :

$$c_k = \frac{a_k - jb_k}{2} \qquad c_{-k} = \frac{a_k + jb_k}{2}$$

From these equations, we know that $c_k$ and $c_{-k}$ must be **complex conjugates** for the signal to be real.

$$c_k = c^*_{-k}$$

**Alternate explanation**:

Notice that the eulers identity can be rewritten as

$$A\cos(x) = \frac{A}{2}\left(e^{jx} + e^{-jx}\right)$$

This shows that each cosine terms need two complex exponentials. This is why the series goes from $-N$ to $N$, which is the same as the original going from 1 to $N$

> Question: If $x(t)$ is periodic with period of 1, can we write
>
> $$x(t) = \sum_{k=-N}^{N} e_k e^{j2\pi kt}$$

Answer: not in a finite $N$. In fact, we can contain large informations with small number of sampling so we dont have to sample near infinity. "Its close enough"

## §5.3 Plotting Fourier Coefficients

### §5.3.1 Finding $c_k$

**Question**: How do we derive $c_k$?

In our case where the fundamental period is 1, lets say we can write our signal in summations of cosines:

$$x(t) = \sum_{k=-N}^{N} c_k e^{j2\pi kt}$$

We want to isolate a given $c_k$ value, only focusing on the $m$th term. Rearranging,

$$c_m e^{j2\pi mt} = x(t) - \sum_{k\neq m} c_k e^{j2\pi kt}$$

Multiplying both sides by $e^{-2\pi mt}$

$$c_m = x(t)e^{-j2\pi mt} - \sum_{k\neq m} c_k e^{j2\pi kt}e^{-j2\pi mt}$$

This still didn't simplify the equation. Now, we can integrate over one period of the signal (0 to 1 in this case):

$$\int_0^1 c_m dt = \int_0^1 x(t)e^{-j2\pi mt}dt - \sum_{k\neq m} c_k \int_0^1 e^{j2\pi(k-m)t}dt$$

How does this integral help? From the left, we just have $\int_0^1 c_m dt = c_m$. The right side, if given the $x(t)$, we can simplify by evaluating the integral of $x(t)$. The last part, we evaluate the integral:

$$c_m = \int_0^1 x(t)e^{-j2\pi mt}dt - \sum_{k\neq m} c_k \left[\frac{1}{j2\pi(k-m)}e^{2\pi(k-m)t}|_0^1\right]$$

Since $k$ and $m$ are integers, $e^{j2\pi(k-m)}$ is always 1 (1 along the real line rotated by multiples of $2\pi$). Then,

$$\frac{1}{j2\pi(k-m)}\left(e^{j2\pi(k-m)} - 1\right) = 0$$

Also, since $k \neq m$, there is not gonna be any indeterminate form. The final form is

$$\boxed{c_k = \int_0^1 x(t)e^{-j2\pi kt}dt}$$

Since $c_k$ is a complex value, $c_k$ has an amplitude and a phase and can be written as

$$c_k = |c_k|e^{j\measuredangle c_k}$$

This is why when we plot $c_k$, we have two graphs

- Amplitude: $|c_k|$ vs $k$
- Phase: $\measuredangle c_k$ vs $k$

The coefficient $k$ corresponds to the frequency, since frequencies are integer multiples of the fundamental

$$f = kf_0$$

**Note**: The bound of the integral is from 0 to 1 because we are integrating over a single period.

**Note**: Any non-smooth function needs infinitely many coefficients to be exact. Smooth mathematically means infinitely differentiable.

### §5.3.2 General Fourier Series

**Definition General Fourier Series**
$x(t)$ is periodic with a period $T_0$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$

where $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

For a particular $c_m$,

$$c_m = x(t)e^{-jm\omega_0 t} - \sum_{k\neq m} c_k e^{jk\omega_0 t} e^{-jm\omega_0 (k-m)t} dt$$

Integrating over a period,

$$\int_0^{T_0} c_m dt = \int_0^{T_0} x(t)e^{-jm\omega_0 t} dt - \sum_{k\neq m} c_k \int_0^{T_0} e^{j\omega_0 (k-m)t} dt$$

Here, we want the term

$$c_k \frac{1}{j\omega_0 (k-m)} \left[ e^{j\omega_0 (k-m)T_0} - 1 \right]$$

to be zero.

$$e^{j\frac{2\pi}{T_0}(k-m)T_0 = e^{j2\pi(k-m)}} = 1$$

The general form for **analysis** (dividing signals) is

$$c_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

and the general form for **syntheis** is

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

### §5.3.3 Plotting FS Coefficients

Coefficnets $c_k$ are **complex** so we can plot $|c_k|$ and $\measuredangle c_k$ separately

- for $|c_k|$, the graph is symmetric to y-axis
- for the phase, there is a symmetry with the origin

## §5.4 Plotting Fourier Coefficients

Goal: for a given signal, lets plot a fourier series

For building intuition, we are just going to build fouier series using sinusoids. Lets say

$$\sum_{k=0}^{\infty} c_k x^k = 5x + 10x^3 - 2x^{10}$$

Then,

$$c_1 = 5$$
$$c_3 = 10$$
$$c_{10}^{=-2}$$

We can do the same (matching terms) with the summation of cosines

**Example:** Consider a signal

$$x(t) = \cos(2\pi 2t)$$

How can we plot the fourier series graph?

The *fundamental frequency* in this case is

$$f_0 = 2\text{Hz} \qquad \omega_0 = 2\pi f_0 = 4\pi(rad/s)$$

We can decompose signals using euler's identity

$$\begin{cases} \cos(x) & = \frac{1}{2}\left(e^{jx} + e^{-jx}\right) \\ \sin(x) & = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right) \end{cases}$$

Decomposing our wave $x(t)$, we get

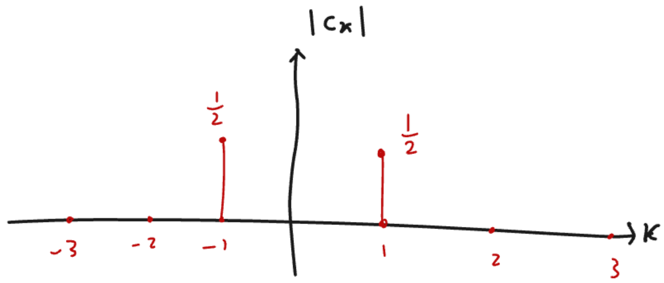
$$x(t) = \frac{1}{2}\left(e^{j2\pi 2t} + e^{-j2\pi 2t}\right)$$

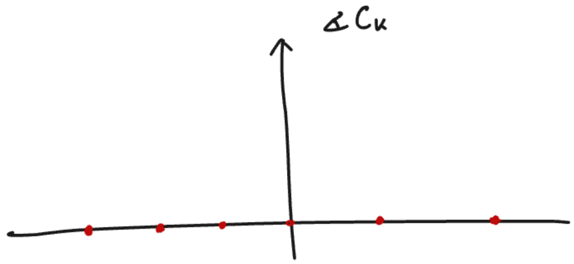
This is equal to the entire fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t} = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{j(-1)\omega_0 t}$$

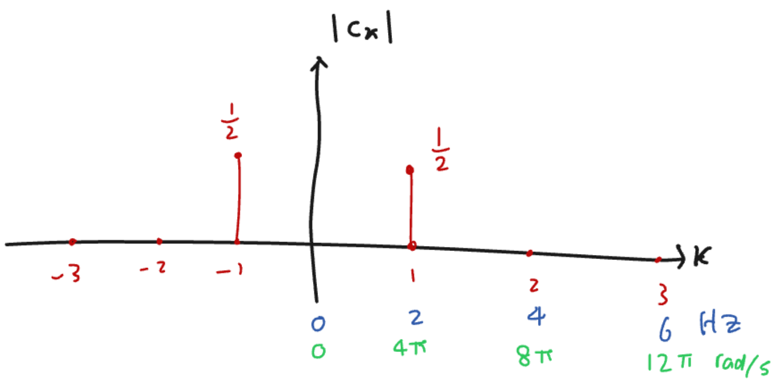
We can start matching terms, and we know that $c_1$ and $c_{-1}$ will have nonzero terms.

$$c_1 = \frac{1}{2}$$
$$c_{-1} = \frac{1}{2}$$




The graph shows us that the only frequency we have is one times the fundamental. We can also represent the graphs in terms of $f$ and $\omega$:




Each $k$ is the multiple of the fundamentals.

**Example:** Consider a signal

$$x(t) = \cos(2\pi 2t) + 2\cos(2\pi 3t - \frac{\pi}{3})$$

*Solution.* Question: is the signal periodic? We can list frequencies of both waves

$$f_1 = 2, T_1 = \frac{1}{2} \qquad f_2 = 3, T_2 = \frac{1}{3}$$

We can find the fundamental frequency using the **GCD**

$$f_0 = \gcd(f_1, f_2) = 1 \text{ Hz}$$

Now we can decompose the signal:

$$= \frac{1}{2}\left(e^{j2\pi 2t} + e^{-j2\pi 2t}\right) + \left(e^{j\left(2\pi 3t - \frac{\pi}{3}\right)} + e^{-j\left(2\pi 3t - \frac{\pi}{3}\right)}\right)$$

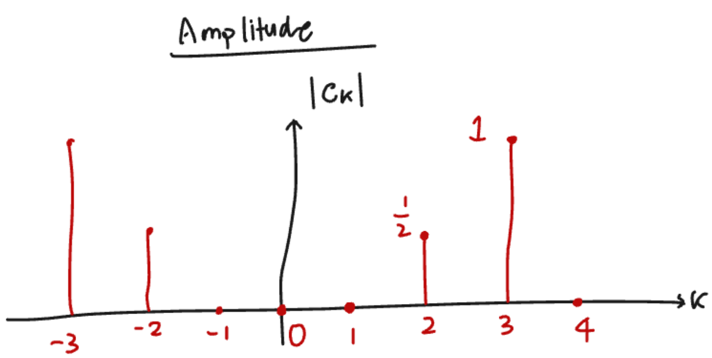
We can match terms with the fourier series

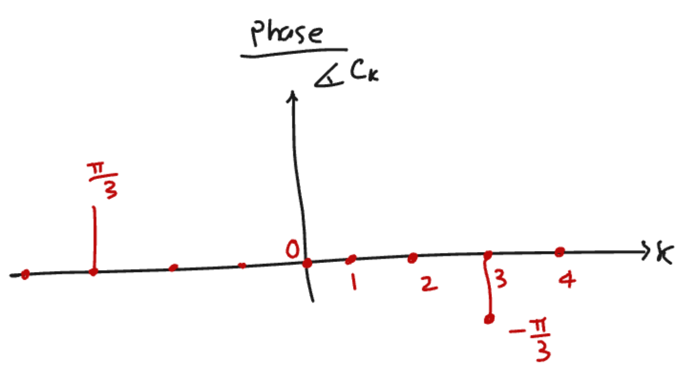
$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

The first term $\frac{1}{2}e^{j4\pi t}$ corresponds to $c_2$ since $\omega_2 = 4\pi$. In a similar sense, we can match

$$c_2 = \frac{1}{2}$$
$$c_{-2} = \frac{1}{2}$$
$$c_3 = e^{j\pi/3}$$
$$c_{-3} = e^{j\pi/3}$$

We can now plot the amplitude and the phase



Remember that amplitude has a symmetry across the y-axis and the phase is symmetric across the origin.

□

**Example:** Consider a signal

$$x(t) = 5 - 3\cos(100t) + 8\sin(300t + \frac{3\pi}{4})$$

*Solution.* Our $\omega_0$ is going to be the greatest common divisor between $\omega_1$ and $\omega_2$ :

$$\omega_0 = \gcd(100, 300) = 100$$

The nonzero coefficients are going to be $0, 1, 2, -1, -2$

We can convert all the negative cosine terms into positive using a phase shift $\cos(x) = \cos(2\pi - x)$. We can also convert the sine term into a cosine by using $\cos(x - \frac{\pi}{2}) = \sin(x)$ Then,

$$\begin{aligned} x(t) &= 5 + 3\cos(100t - pi) + 8\cos\left(300t + \frac{3\pi}{4} - \frac{\pi}{2}\right) \\ &= 5 + \frac{3}{2}\left(e^{j(100t-\pi)} + e^{-j(100t-\pi)}\right) + \frac{8}{2}\left(e^{j(300t+\frac{\pi}{4})} + e^{-j\left(300t+\frac{\pi}{4}\right)}\right) \\ &= 5 + \frac{3}{2}e^{-j\pi}e^{j100t} + \frac{3}{2}e^{j\pi}e^{-j100t} + 4e^{j\pi/4}e^{j300t} + 4e^{jj\pi/4}e^{-j300t} \end{aligned}$$

We can match each term with its frequency

$$c_0 = 5 \qquad c_1 = c_{-1} = \frac{3}{2} \qquad c_3 = c_{-3} = 4$$

We do not have $c_2$ because we had nothing with two times the fundamental frequency $f_0$. We only had 100 and 300, not $\omega = 200$.

□

## §5.5 Fourier Transform

Fourier transform is used to find the frequency content of aperiodic signals. This can be viewed as a foruier series where $T_0 \to \infty$.

Assume a aperiodic signal from $-T_1$ to $T_1$.

We can periodize the signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

From our derivation from the previous lecture,

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x} e^{jk\omega_0 t} dt = \frac{1}{T_0}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$

Instead of having a continuous $\omega$, we can sample using

$$X(\omega) = \inf_{-\infty}^{\infty} x(t)ee^{-j\omega t}dt$$

We can rewrite the series as

$$\sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(k\omega_0)e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0)e^{jk\omega_0 t}$$

As $T_0$ approaches infinity, then the $\tilde{x}$ approaches $x(t)$ since we are pushing waves away from each other. Then, $\omega_0 \to 0$. Effectively, we have made a riemman sum.

**Definition Fourier Transform**
Fourier transform for analysis:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier transform for synthesis:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Why fourier transform? Fourier analysis only can analyze with multiples of the fundamental, so if we stop, we would be looking at the periotized signal instead of the aperiodic signal.

## §5.6 Sampling

**Definition Sampling**
We want to turn a continuous signal into a discrete signal

$$x(t) \to x[n]$$

where $n \in \mathbb{Z}$

There are nonuniform sampling, but we stick with uniform sampling for this course.

**Example:** Consider a signal

$$x(t) = \cos(2\pi f_0 t + \phi_0)$$

Sample at $f_s$

We get

$$x(t) = \cos(2\pi f_0 t + \phi_0) \to x[n] = x[nT_s] = x\left(\frac{n}{f_s}\right)$$

$$= \cos(2\pi f_0 \frac{n}{f_s} + \phi_0 = \cos\left(2\pi \left(\frac{f_0}{f_s}\right) n + \phi_0\right)$$

We consider another frequency

$$y(t) = \cos(2\pi(f_0 + f_s)t + \phi_0) \Rightarrow y[n] = y\left(\frac{n}{f_s}\right)$$

$$= \cos(2\pi\left(f_0 + f_s\right)\left(\frac{n}{f_s} + \phi_0\right) = \cos\left(2\pi\frac{f_0}{f_s}n + 2\pi\left(\frac{f_s}{f_s}n + \phi_0\right)\right) = x\left[n\right]$$

## §5.6.1 Aliasing

Lets say we have two signals

$$x_1(t) = \cos(400\pi t) \qquad x_2(t) = \cos(2400\pi t)$$

where they are sampled at $f_s = 1000$ Hz.

$$x_1\left[n\right] = \cos\left(400\pi\left(\frac{n}{1000}\right)\right) = \cos\left(0.4\pi n\right)$$
$$x_2\left[n\right] = \cos\left(2400\pi\left(\frac{n}{1000}\right)\right) = \cos(0.4\pi n) = x_1\left[n\right]$$

This is a problem because $x_1$ and $x_2$ are different waves. In fact, $\cos(4.4\pi n)$, $\cos(6.4\pi n)$,... would look the same.

**Definition Normalized Frequency**
The normalized frequency $\hat{\omega}$ is

$$\hat{\omega} = \frac{\omega}{f_s}$$

## §5.6.2 Reconstruction

In whatever frequency we sample, there are *infinitely many* signals that goes through all of the points (not necessarily sinusoidal)

If we want to go back to a continuous time signal, choose the smoothest signal that goes through the samples. Essentially, smoothest = lowest frequency.

Since the normalized frequency spectrum is $2\pi \cdot$ periodic, we will choose the range between $-\pi$ and $\pi$ to preserve symmetry.

**How to determine aliasing**

1. normalize frequency spectrum $f_s$ to get $\hat{\omega}$
2. plot spectrum as a function of $\hat{\omega}$ (pretend this is the only frequency we have)
3. Periodize spectrum by $2\pi$
4. Take whatever ends up in the range $[-\pi, \pi]$

if reconstructed signal in $[-\pi, \pi]$ is **not** same as the original, **aliasing** has occured.

**Example:** For sinusoids, aliasing is going to result in a lower frequency.

$$x(t) = \cos(2400\pi t), f_s = 1000Hz$$

We solve for normalized frequency:

$$\hat{\omega}_1 = \frac{2400\pi}{f_s} = 2.4\pi$$

We then plot the spectrum:

(plot)

Proof for all general signals:

We can think of any general signal a combination of cosine waves. Lets say we have a general signal $x(t)$ and perform a fourier transform. After sampling,

**Theorem 5.6.5 Shannon's Sampling Theorem**
A continuous time signal $x(t)$ with frequencies no greater than $f_m$ can be reconstructed exactly from its samples if $f_s > 2f_m$

*Proof.* Revisit proof from 11/7 lecture $\square$